Ch. 6 POWER FLOWS

Note Title 4/27/2014

6.1 Direct Solutions to Linear Algebraic Equations: Gauss Elimination

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & & \vdots & \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \implies \mathbf{A}\mathbf{x} = \mathbf{y}$$

Steps:

1) Transform A into uppar triangular matrix.

$$\begin{bmatrix} A_{11} & A_{12} \ldots & & A_{1N} \\ 0 & A_{22} \ldots & & A_{2N} \\ \vdots & & & \\ 0 & 0 \ldots & A_{N-1,N-1} & A_{N-1,N} \\ 0 & 0 \ldots 0 & & A_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \\ y_N \end{bmatrix}$$

How? <u>Gauss elimination</u>

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ 0 & \left(A_{22} - \frac{A_{21}}{A_{11}} A_{12}\right) & \cdots & \left(A_{2N} - \frac{A_{21}}{A_{11}} A_{1N}\right) \\ 0 & \left(A_{32} - \frac{A_{31}}{A_{11}} A_{12}\right) & \cdots & \left(A_{3N} - \frac{A_{31}}{A_{11}} A_{1N}\right) \\ \vdots & \vdots & & \vdots \\ 0 & \left(A_{N2} - \frac{A_{N1}}{A_{11}} A_{12}\right) & \cdots & \left(A_{NN} - \frac{A_{N1}}{A_{11}} A_{1N}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 - \frac{A_{21}}{A_{11}} y_1 \\ y_3 - \frac{A_{31}}{A_{11}} y_1 \\ \vdots \\ y_N - \frac{A_{N1}}{A_{11}} y_1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} & \cdots & A_{1N}^{(1)} \\ 0 & A_{22}^{(1)} & \cdots & A_{2N}^{(1)} \\ 0 & A_{32}^{(1)} & \cdots & A_{3N}^{(1)} \\ \vdots & \vdots & & \vdots \\ 0 & A_{N2}^{(1)} & \cdots & A_{NN}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} y_1^{(1)} \\ y_2^{(1)} \\ y_3^{(1)} \\ \vdots \\ y_N^{(1)} \end{bmatrix}$$

(1) means first step

$$\Downarrow$$

$$\begin{bmatrix} A_{11}^{(2)} & A_{12}^{(2)} & A_{13}^{(2)} & \cdots & A_{1N}^{(2)} \\ 0 & A_{22}^{(2)} & A_{23}^{(2)} & \cdots & A_{2N}^{(2)} \\ 0 & 0 & A_{33}^{(2)} & \cdots & A_{3N}^{(2)} \\ 0 & 0 & A_{43}^{(2)} & \cdots & A_{4N}^{(2)} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & A_{N3}^{(2)} & \cdots & A_{NN}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} y_1^{(2)} \\ y_2^{(2)} \\ y_3^{(2)} \\ y_4^{(2)} \\ \vdots \\ y_N^{(2)} \end{bmatrix}$$

2) Find the values of x_k using the back substitition method:

$$\begin{bmatrix} A_{11} & A_{12}\ldots & & A_{1N} \\ 0 & A_{22}\ldots & & A_{2N} \\ \vdots & & & \\ 0 & 0\ldots & A_{N-1,N-1} & A_{N-1,N} \\ 0 & 0\ldots 0 & & A_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \\ y_N \end{bmatrix}$$

start from the last element to get:

$$x_N = \frac{y_N}{A_{NN}}$$

then the next-to-last element:

$$x_{N-1} = \frac{y_{N-1} - A_{N-1,N}x_N}{A_{N-1,N-1}}$$

In general

$$x_k = \frac{y_k - \sum_{n=k+1}^{N} A_{kn}x_n}{A_{kk}} \qquad k = N, N-1, \ldots, 1$$

EXAMPLE 6.1 Gauss elimination and back substitution: direct solution to linear algebraic equations

Solve

$$\left[\begin{array}{c|c} 10 & 5 \\ \hline 2 & 9 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

using Gauss elimination and back substitution.

SOLUTION Since $N = 2$ for this example, there is $(N-1) = 1$ Gauss elimination step. Multiplying the first equation by $A_{21}/A_{11} = 2/10$ and then subtracting from the second,

$$\left[\begin{array}{c|c} 10 & 5 \\ \hline 0 & 9 - \frac{2}{10}(5) \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 - \frac{2}{10}(6) \end{bmatrix}$$

or

$$\left[\begin{array}{c|c} 10 & 5 \\ \hline 0 & 8 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1.8 \end{bmatrix}$$

which has the form $\mathbf{A}^{(1)}\mathbf{x} = \mathbf{y}^{(1)}$, where $\mathbf{A}^{(1)}$ is upper triangular. Now, using back substitution, (6.1.6) gives, for $k = 2$:

$$x_2 = \frac{y_2^{(1)}}{A_{22}^{(1)}} = \frac{1.8}{8} = 0.225$$

and, for $k = 1$,

$$x_1 = \frac{y_1^{(1)} - A_{12}^{(1)}x_2}{A_{11}^{(1)}} = \frac{6 - (5)(0.225)}{10} = 0.4875 \quad \blacksquare$$

read Ex 6.2

6.2 Iterative Solutions to Linear Algebraic Equations: Jacobi and Gauss-Seidel:

* Jacobi method:

$$y_k = A_{k1}x_1 + A_{k2}x_2 + \cdots + A_{kk}\boxed{x_k} + \cdots + A_{kN}x_N$$

solving for x_k

$$x_k = \frac{1}{A_{kk}}[y_k - (A_{k1}x_1 + \cdots + A_{k,k-1}x_{k-1} + A_{k,k+1}x_{k+1} + \cdots + A_{kN}x_N)]$$

$$= \frac{1}{A_{kk}}\left[y_k - \sum_{n=1}^{k-1} A_{kn}x_n - \sum_{n=k+1}^{N} A_{kn}x_n\right]$$

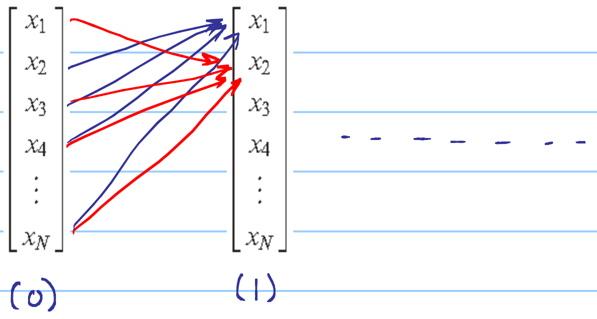
steps:

1) assume initial values for $x_k \Rightarrow x_k(0)$

2) $$x_k(i+1) = \frac{1}{A_{kk}}\left[y_k - \sum_{n=1}^{k-1} A_{kn}x_n(i) - \sum_{n=k+1}^{N} A_{kn}x_n(i)\right] \qquad k = 1, 2, \ldots, N$$

3) Stop when $$\left|\frac{x_k(i+1) - x_k(i)}{x_k(i)}\right| < \varepsilon \qquad \text{for all } k = 1, 2, \ldots, N$$

tolerance level



Solving in matrix form:

$$\mathbf{x}(i+1) = \mathbf{M}\mathbf{x}(i) + \mathbf{D}^{-1}\mathbf{y}$$

where

$$\mathbf{M} = \mathbf{D}^{-1}(\mathbf{D} - \mathbf{A})$$

and

$$\mathbf{D} = \begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ 0 & A_{22} & 0 & \cdots & 0 \\ 0 & \vdots & \vdots & & \vdots \\ \vdots & & & & 0 \\ 0 & 0 & 0 & \cdots & A_{NN} \end{bmatrix}$$

EXAMPLE 6.3 Jacobi method: iterative solution to linear algebraic equations

Solve Example 6.1 using the Jacobi method. Start with $x_1(0) = x_2(0) = 0$ and continue until (6.2.2) is satisfied for $\varepsilon = 10^{-4}$.

SOLUTION From (6.2.5) with $N = 2$,

$$k = 1 \qquad x_1(i+1) = \frac{1}{A_{11}}[y_1 - A_{12}x_2(i)] = \frac{1}{10}[6 - 5x_2(i)]$$

$$k = 2 \qquad x_2(i+1) = \frac{1}{A_{22}}[y_2 - A_{21}x_1(i)] = \frac{1}{9}[3 - 2x_1(i)]$$

Alternatively, in matrix format using (6.2.6)–(6.2.8),

$$\mathbf{D}^{-1} = \left[\begin{array}{c|c} 10 & 0 \\ \hline 0 & 9 \end{array}\right]^{-1} = \left[\begin{array}{c|c} \frac{1}{10} & 0 \\ \hline 0 & \frac{1}{9} \end{array}\right]$$

$$\mathbf{M} = \left[\begin{array}{c|c} \frac{1}{10} & 0 \\ \hline 0 & \frac{1}{9} \end{array}\right] \left[\begin{array}{c|c} 0 & -5 \\ \hline -2 & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & -\frac{5}{10} \\ \hline -\frac{2}{9} & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1(i+1) \\ x_2(i+1) \end{bmatrix} = \left[\begin{array}{c|c} 0 & -\frac{5}{10} \\ \hline -\frac{2}{9} & 0 \end{array}\right] \begin{bmatrix} x_1(i) \\ x_2(i) \end{bmatrix} + \left[\begin{array}{c|c} \frac{1}{10} & 0 \\ \hline 0 & \frac{1}{9} \end{array}\right] \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

The above two formulations are identical. Starting with $x_1(0) = x_2(0) = 0$, the iterative solution is given in the following table:

JACOBI

i	0	1	2	3	4	5	6	7	8	9	10
$x_1(i)$	0	0.60000	0.43334	0.50000	0.48148	0.48889	0.48683	0.48766	0.48743	0.48752	0.48749
$x_2(i)$	0	0.33333	0.20000	0.23704	0.22222	0.22634	0.22469	0.22515	0.22496	0.22502	0.22500

As shown, the Jacobi method converges to the unique solution obtained in Example 6.1. The convergence criterion is satisfied at the 10th iteration, since

$$\left|\frac{x_1(10) - x_1(9)}{x_1(9)}\right| = \left|\frac{0.48749 - 0.48752}{0.48749}\right| = 6.2 \times 10^{-5} < \varepsilon$$

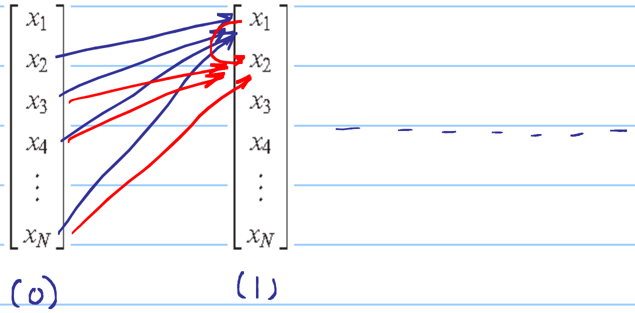
and

$$\left|\frac{x_2(10) - x_2(9)}{x_2(9)}\right| = \left|\frac{0.22500 - 0.22502}{0.22502}\right| = 8.9 \times 10^{-5} < \varepsilon \qquad \blacksquare$$

* Gauss-Seidel method:

the equation in step 2 in the Jacobi method is replaced by this one

$$x_k(i+1) = \frac{1}{A_{kk}}\left[y_k - \sum_{n=1}^{k-1} \mathrm{A}_{kn}x_n(i+1) - \sum_{n=k+1}^{N} \mathrm{A}_{kn}x_n(i)\right]$$



The matrix form is the same:

where,

$$\mathbf{D} = \begin{bmatrix} \mathrm{A}_{11} & 0 & 0 & \cdots & 0 \\ \mathrm{A}_{21} & \mathrm{A}_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ \mathrm{A}_{N1} & \mathrm{A}_{N2} & \cdots & & \mathrm{A}_{NN} \end{bmatrix}$$

EXAMPLE 6.4 **Gauss–Seidel method: iterative solution to linear algebraic equations**

Rework Example 6.3 using the Gauss–Seidel method.

SOLUTION From (6.2.9),

$$k = 1 \qquad x_1(i+1) = \frac{1}{\mathrm{A}_{11}}[y_1 - \mathrm{A}_{12}x_2(i)] = \frac{1}{10}[6 - 5x_2(i)]$$

$$k = 2 \qquad x_2(i+1) = \frac{1}{\mathrm{A}_{22}}[y_2 - \mathrm{A}_{21}x_1(i+1)] = \frac{1}{9}[3 - 2x_1(i+1)]$$

Using this equation for $x_1(i+1)$, $x_2(i+1)$ can also be written as

$$x_2(i+1) = \frac{1}{9}\left\{3 - \frac{2}{10}[6 - 5x_2(i)]\right\}$$

Alternatively, in matrix format, using (6.2.10), (6.2.6), and (6.2.7):

$$\mathbf{D}^{-1} = \left[\begin{array}{c|c} 10 & 0 \\ 2 & 9 \end{array}\right]^{-1} = \left[\begin{array}{c|c} \frac{1}{10} & 0 \\ \hline -\frac{2}{90} & \frac{1}{9} \end{array}\right]$$

$$\mathbf{M} = \left[\begin{array}{c|c} \frac{1}{10} & 0 \\ \hline -\frac{2}{90} & \frac{1}{9} \end{array}\right] \left[\begin{array}{c|c} 0 & -5 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & -\frac{1}{2} \\ \hline 0 & \frac{1}{9} \end{array}\right]$$

$$\begin{bmatrix} x_1(i+1) \\ x_2(i+1) \end{bmatrix} = \left[\begin{array}{c|c} 0 & -\frac{1}{2} \\ \hline 0 & \frac{1}{9} \end{array}\right] \begin{bmatrix} x_1(i) \\ x_2(i) \end{bmatrix} + \left[\begin{array}{c|c} \frac{1}{10} & 0 \\ \hline -\frac{2}{90} & \frac{1}{9} \end{array}\right] \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

These two formulations are identical. Starting with $x_1(0) = x_2(0) = 0$, the solution is given in the following table:

GAUSS–SEIDEL

i	0	1	2	3	4	5	6
$x_1(i)$	0	0.60000	0.50000	0.48889	0.48765	0.48752	0.48750
$x_2(i)$	0	0.20000	0.22222	0.22469	0.22497	0.22500	0.22500

For this example, Gauss–Seidel converges in 6 iterations, compared to 10 iterations with Jacobi. ■

6.3 Iterative Solutions to Non-linear Algebraic Equations: Newton-Raphson:

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_N(\mathbf{x}) \end{bmatrix} = \mathbf{y} \implies \mathbf{0} = \mathbf{y} - \mathbf{f}(\mathbf{x})$$

$$\mathbf{x}(i+1) = \mathbf{x}(i) + \mathbf{D}^{-1}\{\mathbf{y} - \mathbf{f}[\mathbf{x}(i)]\} \rightarrow \text{extended Gauss-Seidel method}$$

For linear equations, $\mathbf{f}(\mathbf{x}) = \mathbf{Ax}$ and (6.3.5) reduces to

$$\mathbf{x}(i+1) = \mathbf{x}(i) + \mathbf{D}^{-1}[\mathbf{y} - \mathbf{Ax}(i)] = \mathbf{D}^{-1}(\mathbf{D} - \mathbf{A})\mathbf{x}(i) + \mathbf{D}^{-1}\mathbf{y} \quad \text{(like Gauss-Seidel method)}$$

Newton- Raphson method:

$$\mathbf{x}(i+1) = \mathbf{x}(i) + \mathbf{J}^{-1}(i)\{\mathbf{y} - \mathbf{f}[\mathbf{x}(i)]\}$$

where

$$\mathbf{J}(i) = \left.\frac{d\mathbf{f}}{d\mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}(i)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_N} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_N}{\partial x_1} & \frac{\partial f_N}{\partial x_2} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix}_{\mathbf{x}=\mathbf{x}(i)}$$

(Jacobian matrix)

Read Ex. 6.6

EXAMPLE 6.7 **Newton–Raphson method: solution to nonlinear algebraic equations**

Solve

$$\begin{bmatrix} x_1 + x_2 \\ x_1 x_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 50 \end{bmatrix} \qquad \mathbf{x}(0) = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$$

Use the Newton–Raphson method starting with the above $\mathbf{x}(0)$ and continue until (6.2.2) is satisfied with $\varepsilon = 10^{-4}$.

SOLUTION Using (6.3.10) with $f_1 = (x_1 + x_2)$ and $f_2 = x_1 x_2$,

$$\mathbf{J}(i)^{-1} = \left[\begin{array}{c|c} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \hline \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{array}\right]^{-1}_{\mathbf{x}=\mathbf{x}(i)} = \left[\begin{array}{c|c} 1 & 1 \\ \hline x_2(i) & x_1(i) \end{array}\right]^{-1} = \frac{\left[\begin{array}{c|c} x_1(i) & -1 \\ \hline -x_2(i) & 1 \end{array}\right]}{x_1(i) - x_2(i)}$$

Using $\mathbf{J}(i)^{-1}$ in (6.3.9),

$$\begin{bmatrix} x_1(i+1) \\ x_2(i+1) \end{bmatrix} = \begin{bmatrix} x_1(i) \\ x_2(i) \end{bmatrix} + \frac{\left[\begin{array}{c|c} x_1(i) & -1 \\ \hline -x_2(i) & 1 \end{array}\right]}{x_1(i) - x_2(i)} \begin{bmatrix} 15 - x_1(i) - x_2(i) \\ 50 - x_1(i)x_2(i) \end{bmatrix}$$

Writing the preceding as two separate equations,

$$x_1(i+1) = x_1(i) + \frac{x_1(i)[15 - x_1(i) - x_2(i)] - [50 - x_1(i)x_2(i)]}{x_1(i) - x_2(i)}$$

$$x_2(i+1) = x_2(i) + \frac{-x_2(i)[15 - x_1(i) - x_2(i)] + [50 - x_1(i)x_2(i)]}{x_1(i) - x_2(i)}$$

Successive calculations of these equations are shown in the following table:

NEWTON–RAPHSON

i	0	1	2	3	4
$x_1(i)$	4	5.20000	4.99130	4.99998	5.00000
$x_2(i)$	9	9.80000	10.00870	10.00002	10.00000

Newton–Raphson converges in four iterations for this example. ■

Calculation of J^{-1} can be replaced by the follwing steps:

$$\mathbf{J}(i)\Delta\mathbf{x}(i) = \Delta\mathbf{y}(i)$$

where

$$\Delta\mathbf{x}(i) = \mathbf{x}(i+1) - \mathbf{x}(i)$$

and

$$\Delta\mathbf{y}(i) = \mathbf{y} - \mathbf{f}[\mathbf{x}(i)]$$

Then, during each iteration, the following four steps are completed:

STEP 1 Compute $\Delta\mathbf{y}(i)$ from (6.3.13).

STEP 2 Compute $\mathbf{J}(i)$ from (6.3.10).

STEP 3 Using Gauss elimination and back substitution, solve for $\Delta\mathbf{x}(i)$.

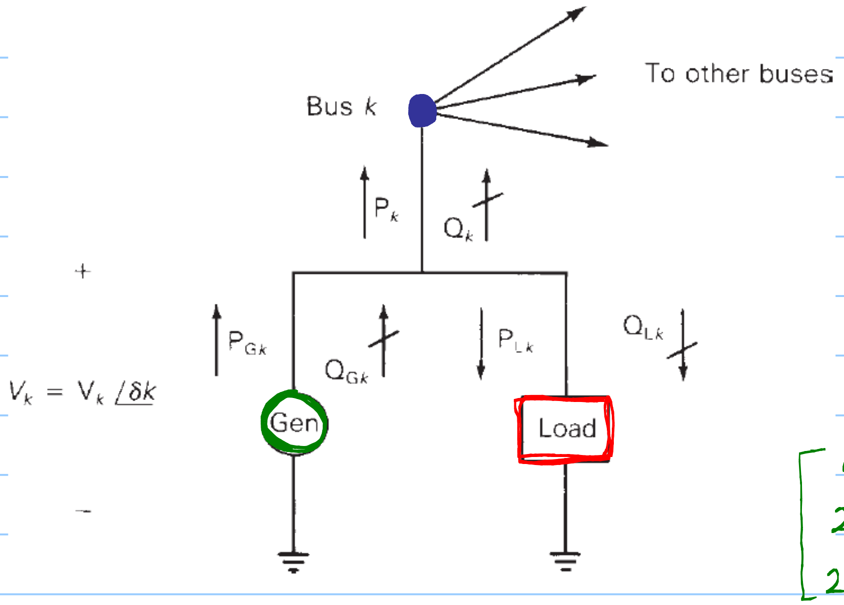
STEP 4 Compute $\mathbf{x}(i+1)$ from (6.3.12).

Read Ex 6.8

6.4 The Power-Flow Problems:

PF problems ⇒ $\underline{V \angle \delta}$ at each bus.
also P & Q can be computed.

* PF problems start with singl-line diagram with the following input data:
① Bus data
② Transmission data
③ Transformer data.



Bus variables:
① V_k
② δ_k
③ $P_k = P_{Gk} - P_{Lk}$
④ $Q_k = Q_{Gk} - Q_{Lk}$

at each bus:
2 knowns (input)
2 unknowns

Each bus k is categorized into one of the following three bus types:

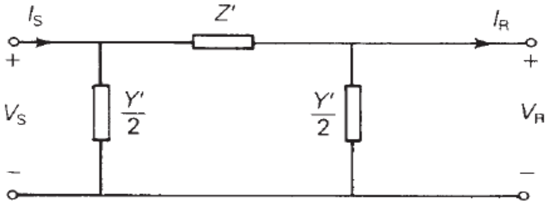
1. Swing bus (or slack bus)—There is only one swing bus, which for convenience is numbered bus 1 in this text. The swing bus is a reference bus for which $V_1 \angle \delta_1$, typically $1.0\angle 0°$ per unit, is input data. The power-flow program computes P_1 and Q_1.

2. Load (PQ) bus—P_k and Q_k are input data. The power-flow program computes V_k and δ_k. Most buses in a typical power-flow program are load buses.

3. Voltage controlled (PV) bus—P_k and V_k are input data. The power-flow program computes Q_k and δ_k. Examples are buses to which generators, switched shunt capacitors, or static var systems are connected. Maximum and minimum var limits $Q_{Gk\max}$ and $Q_{Gk\min}$ that this equipment can supply are also input data. If an upper or lower reactive power limit is reached, then the reactive power output of the generator is held at the limit, and the bus is modeled as a PQ bus. Another example is a bus to which a tap-changing transformer is connected; the power-flow program then computes the tap setting.

Transmission lines and Transformers Representation (in pu)

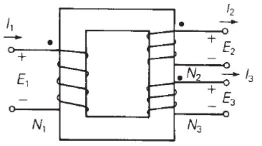
Transmission lines



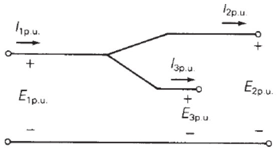
$$Z' = Z_c \sinh(\gamma\ell) = ZF_1 = Z\frac{\sinh(\gamma\ell)}{\gamma\ell}$$

$$\frac{Y'}{2} = \frac{\tanh(\gamma\ell/2)}{Z_c} = \frac{Y}{2}F_2 = \frac{Y}{2}\frac{\tanh(\gamma\ell/2)}{(\gamma\ell/2)}$$

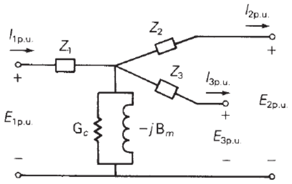
Transformers (Three-winding)



(a) Basic core and coil configuration



(b) Per-unit equivalent circuit—ideal transformer



(c) Per-unit equivalent circuit—practical transformer

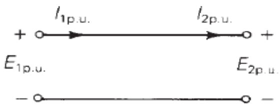
The bus admittance matrix Y_{bus} can be constructed from the line and transformer input data. From (2.4.3) and (2.4.4), the elements of Y_{bus} are:

Diagonal elements: Y_{kk} = sum of admittances connected to bus k

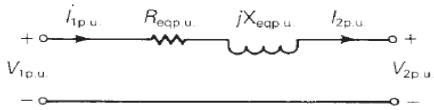
Off-diagonal elements: $Y_{kn} = -$(sum of admittances connected between buses k and n)

$$k \neq n \qquad (6.4.2)$$

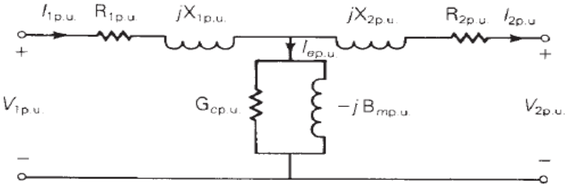
Transformers (Two-winding)



(a) Ideal transformer

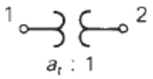


(b) Neglecting exciting current

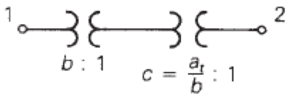


(c) Complete representation

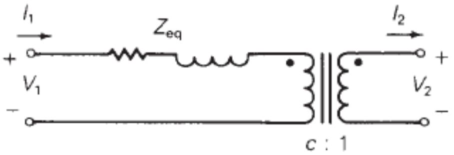
Transformer (Tap-changing)



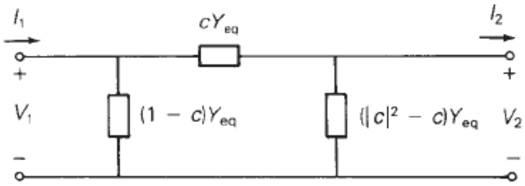
(a) Single-line diagram



(b) Represented as two transformers in series



(c) Per-unit equivalent circuit
(Per-unit impedance is shown)



(d) π circuit representation for real c
(Per-unit admittances are shown; $Y_{eq} = \frac{1}{Z_{eq}}$)

EXAMPLE 6.9 Power-flow input data and $\boldsymbol{Y}_{\text{bus}}$

Figure 6.2 shows a single-line diagram of a five-bus power system. Input data are given in Tables 6.1, 6.2, and 6.3. As shown in Table 6.1, bus 1, to which a generator is connected, is the swing bus. Bus 3, to which a generator and a load are connected, is a voltage-controlled bus. Buses 2, 4, and 5 are load buses. Note that the loads at buses 2 and 3 are inductive since $Q_2 = -Q_{L2} = -2.8$ and $-Q_{L3} = -0.4$ are negative.

For each bus k, determine which of the variables V_k, δ_k, P_k, and Q_k are input data and which are unknowns. Also, compute the elements of the second row of $\boldsymbol{Y}_{\text{bus}}$.

SOLUTION The input data and unknowns are listed in Table 6.4. For bus 1, the swing bus, P_1 and Q_1 are unknowns. For bus 3, a voltage-controlled bus,

FIGURE 6.2

Single-line diagram for Example 6.9

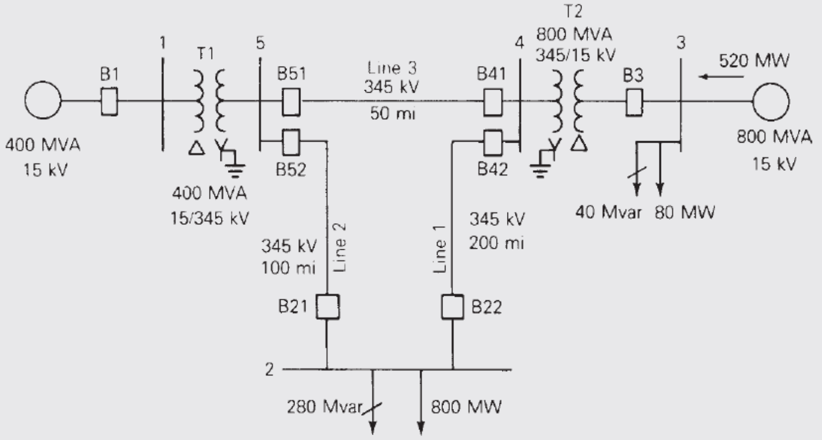


TABLE 6.1

Bus input data for Example 6.9*

Bus	Type	V per unit	δ degrees	P_G per unit	Q_G per unit	P_L per unit	Q_L per unit	Q_{Gmax} per unit	Q_{Gmin} per unit
1	Swing	1.0	0			0	0		
2	Load			0	0	8.0	2.8		
3	Constant voltage	1.05		5.2		0.8	0.4	4.0	−2.8
4	Load			0	0	0	0		
5	Load			0	0	0	0		

* $S_{\text{base}} = 100$ MVA, $V_{\text{base}} = 15$ kV at buses 1, 3, and 345 kV at buses 2, 4, 5

TABLE 6.2

Line input data for Example 6.9

Bus-to-Bus	R′ per unit	X′ per unit	G′ per unit	B′ per unit	Maximum MVA per unit
2–4	0.0090	0.100	0	1.72	12.0
2–5	0.0045	0.050	0	0.88	12.0
4–5	0.00225	0.025	0	0.44	12.0

TABLE 6.3

Transformer input data for Example 6.9

Bus-to-Bus	R per unit	X per unit	G_c per unit	B_m per unit	Maximum MVA per unit	Maximum TAP Setting per unit
1–5	0.00150	0.02	0	0	6.0	
3–4	0.00075	0.01	0	0	10.0	

TABLE 6.4

Input data and unknowns for Example 6.9

Bus	Input Data	Unknowns
1	$V_1 = 1.0, \delta_1 = 0$	P_1, Q_1
2	$P_2 = P_{G2} - P_{L2} = -8$ $Q_2 = Q_{G2} - Q_{L2} = -2.8$	V_2, δ_2
3	$V_3 = 1.05$ $P_3 = P_{G3} - P_{L3} = 4.4$	Q_3, δ_3
4	$P_4 = 0, Q_4 = 0$	V_4, δ_4
5	$P_5 = 0, Q_5 = 0$	V_5, δ_5

Q_3 and δ_3 are unknowns. For buses 2, 4, and 5, load buses, V_2, V_4, V_5 and δ_2, δ_4, δ_5 are unknowns.

The elements of $\boldsymbol{Y}_{\text{bus}}$ are computed from (6.4.2). Since buses 1 and 3 are not directly connected to bus 2,

$$Y_{21} = Y_{23} = 0$$

Using (6.4.2),

$$Y_{24} = \frac{-1}{R'_{24} + jX'_{24}} = \frac{-1}{0.009 + j0.1} = -0.89276 + j9.91964 \quad \text{per unit}$$

$$= 9.95972\underline{/95.143^\circ} \quad \text{per unit}$$

$$Y_{25} = \frac{-1}{R'_{25} + jX'_{25}} = \frac{-1}{0.0045 + j0.05} = -1.78552 + j19.83932 \quad \text{per unit}$$

$$= 19.9195\underline{/95.143^\circ} \quad \text{per unit}$$

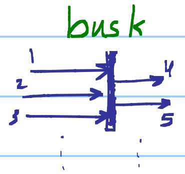
$$Y_{22} = \frac{1}{R'_{24} + jX'_{24}} + \frac{1}{R'_{25} + jX'_{25}} + j\frac{B'_{24}}{2} + j\frac{B'_{25}}{2}$$

$$= (0.89276 - j9.91964) + (1.78552 - j19.83932) + j\frac{1.72}{2} + j\frac{0.88}{2}$$

$$= 2.67828 - j28.4590 = 28.5847\underline{/-84.624^\circ} \quad \text{per unit}$$

where half of the shunt admittance of each line connected to bus 2 is included in Y_{22} (the other half is located at the other ends of these lines).

$$\boldsymbol{I} = \boldsymbol{Y}_{\text{bus}}\boldsymbol{V} \longrightarrow I_k = \sum_{n=1}^{N} Y_{kn} V_n$$



Complex power:

$$S_k = P_k + jQ_k = V_k I_k^* \Rightarrow P_k + jQ_k = V_k\left[\sum_{n=1}^{N} Y_{kn} V_n\right]^* \qquad k = 1, 2, \ldots, N$$

where,

$$V_n = V_n e^{j\delta_n}$$

$$Y_{kn} = Y_{kn} e^{j\theta_{kn}} = G_{kn} + jB_{kn} \qquad k, n = 1, 2, \ldots, N$$

$$\Rightarrow P_k + jQ_k = V_k \sum_{n=1}^{N} Y_{kn} V_n e^{j(\delta_k - \delta_n - \theta_{kn})}$$

Polar form

$$P_k = V_k \sum_{n=1}^{N} Y_{kn} V_n \cos(\delta_k - \delta_n - \theta_{kn})$$

$$Q_k = V_k \sum_{n=1}^{N} Y_{kn} V_n \sin(\delta_k - \delta_n - \theta_{kn}) \qquad k = 1, 2, \ldots, N$$

OR

Rectangular form

$$P_K = V_K \sum_{n=1}^{N} V_n[G_{kn}\cos(\delta_k - \delta_n) + B_{kn}\sin(\delta_k - \delta_n)] \tag{6.4.12}$$

$$Q_K = V_K \sum_{n=1}^{N} V_n[G_{kn}\sin(\delta_k - \delta_n) - B_{kn}\cos(\delta_k - \delta_n)] \qquad k = 1, 2, \ldots, N \tag{6.4.13}$$

6.5 Power-Flow Solution by Gauss-Seidel :

1) Load Bus:

$$I_k = \frac{P_k - jQ_k}{V_k^*}$$

First

$$V_k(i+1) = \frac{1}{Y_{kk}}\left[\frac{P_k - jQ_k}{V_k^*(i)} - \sum_{n=1}^{k-1} Y_{kn}V_n(i+1) - \sum_{n=k+1}^{N} Y_{kn}V_n(i)\right]$$

then

$$V_k(i+1) = \frac{1}{Y_{kk}}\left[\frac{P_k - jQ_k}{V_k^*(i+1)} - \sum_{n=1}^{k-1} Y_{kn}V_n(i+1) - \sum_{n=k+1}^{N} Y_{kn}V_n(i)\right]$$

2) Voltage - Controlled Bus

$$Q_k = V_k(i)\sum_{n=1}^{N} Y_{kn}V_n(i)\sin[\delta_k(i) - \delta_n(i) - \theta_{kn}]$$

Also,

$$Q_{Gk} = Q_k + Q_{Lk}$$

3) Swing Bus

$$P_k = V_k\sum_{n=1}^{N} Y_{kn}V_n\cos(\delta_k - \delta_n - \theta_{kn})$$

for one iteration only

$$Q_k = V_k\sum_{n=1}^{N} Y_{kn}V_n\sin(\delta_k - \delta_n - \theta_{kn}) \qquad k = 1, 2, \ldots, N$$

EXAMPLE 6.10 Power-flow solution by Gauss–Seidel

For the power system of Example 6.9, use Gauss–Seidel to calculate $V_2(1)$, the phasor voltage at bus 2 after the first iteration. Use zero initial phase angles and 1.0 per-unit initial voltage magnitudes (except at bus 3, where $V_3 = 1.05$) to start the iteration procedure.

SOLUTION Bus 2 is a load bus. Using the input data and bus admittance values from Example 6.9 in (6.5.2),

$$V_2(1) = \frac{1}{Y_{22}}\left\{\frac{P_2 - jQ_2}{V_2^*(0)} - [Y_{21}V_1(1) + Y_{23}V_3(0) + Y_{24}V_4(0) + Y_{25}V_5(0)]\right\}$$

$$= \frac{1}{28.5847\underline{/-84.624^\circ}}\left\{\frac{-8 - j(-2.8)}{1.0\underline{/0^\circ}}\right.$$

$$\left. - [(-1.78552 + j19.83932)(1.0) + (-0.89276 + j9.91964)(1.0)]\right\}$$

$$= \frac{(-8 + j2.8) - (-2.67828 + j29.7589)}{28.5847\underline{/-84.624^\circ}}$$

$$= 0.96132\underline{/-16.543^\circ} \quad \text{per unit}$$

Next, the above value is used in (6.5.2) to recalculate $V_2(1)$:

$$V_2(1) = \frac{1}{28.5847\underline{/-84.624^\circ}}\left\{\frac{-8 + j2.8}{0.96132\underline{/16.543^\circ}}\right.$$

$$\left. - [-2.67828 + j29.75829]\right\}$$

$$= \frac{-4.4698 - j24.5973}{28.5847\underline{/-84.624^\circ}} = 0.87460\underline{/-15.675^\circ} \quad \text{per unit}$$

Computations are next performed at buses 3, 4, and 5 to complete the first Gauss–Seidel iteration.

6.6 Power-Flow Solution by Newton-Raphson:

$$\mathbf{x}=\begin{bmatrix}\boldsymbol{\delta}\\ \mathbf{V}\end{bmatrix}=\begin{bmatrix}\delta_2 \xleftarrow{\text{in radians}}\\ \vdots\\ \delta_N\\ \mathrm{V}_2 \leftarrow \text{in pu}\\ \vdots\\ \mathrm{V}_N\end{bmatrix};\quad \mathbf{y}=\begin{bmatrix}\mathbf{P}\\ \mathbf{Q}\end{bmatrix}=\begin{bmatrix}\mathrm{P}_2 \leftarrow \text{in pu}\\ \vdots\\ \mathrm{P}_N\\ \mathrm{Q}_2 \leftarrow \text{in pu}\\ \vdots\\ \mathrm{Q}_N\end{bmatrix};$$

no swing bus variables $(V_1\angle\underline{\delta_1})$

$$\mathbf{f}(\mathbf{x})=\begin{bmatrix}\mathbf{P}(\mathbf{x})\\ \mathbf{Q}(\mathbf{x})\end{bmatrix}=\begin{bmatrix}\mathrm{P}_2(\mathbf{x})\\ \vdots\\ \mathrm{P}_N(\mathbf{x})\\ \mathrm{Q}_2(\mathbf{x})\\ \vdots\\ \mathrm{Q}_N(\mathbf{x})\end{bmatrix}$$

$$\mathrm{y}_k=\mathrm{P}_k=\mathrm{P}_k(\mathbf{x})=\mathrm{V}_k\sum_{n=1}^{N}\mathrm{Y}_{kn}\mathrm{V}_n\cos(\delta_k-\delta_n-\theta_{kn})$$

$$\mathrm{y}_{k+N}=\mathrm{Q}_k=\mathrm{Q}_k(\mathbf{x})=\mathrm{V}_k\sum_{n=1}^{N}\mathrm{Y}_{kn}\mathrm{V}_n\sin(\delta_k-\delta_n-\theta_{kn})$$

$$k=2,3,\ldots,N$$

The Jacobian matrix of (6.3.10) has the form

$$\mathbf{J}=\left[\begin{array}{ccc|ccc}
 & \mathbf{J1} & & & \mathbf{J2} & \\
\frac{\partial \mathrm{P}_2}{\partial \delta_2} & \cdots & \frac{\partial \mathrm{P}_2}{\partial \delta_N} & \frac{\partial \mathrm{P}_2}{\partial \mathrm{V}_2} & \cdots & \frac{\partial \mathrm{P}_2}{\partial \mathrm{V}_N}\\
\vdots & & & \vdots & & \\
\frac{\partial \mathrm{P}_N}{\partial \delta_2} & \cdots & \frac{\partial \mathrm{P}_N}{\partial \delta_N} & \frac{\partial \mathrm{P}_N}{\partial \mathrm{V}_2} & \cdots & \frac{\partial \mathrm{P}_N}{\partial \mathrm{V}_N}\\
\hline
\frac{\partial \mathrm{Q}_2}{\partial \delta_2} & \cdots & \frac{\partial \mathrm{Q}_2}{\partial \delta_N} & \frac{\partial \mathrm{Q}_2}{\partial \mathrm{V}_2} & \cdots & \frac{\partial \mathrm{Q}_2}{\partial \mathrm{V}_N}\\
\vdots & & & \vdots & & \\
\frac{\partial \mathrm{Q}_N}{\partial \delta_2} & \cdots & \frac{\partial \mathrm{Q}_N}{\partial \delta_N} & \frac{\partial \mathrm{Q}_N}{\partial \mathrm{V}_2} & \cdots & \frac{\partial \mathrm{Q}_N}{\partial \mathrm{V}_N}\\
 & \mathbf{J3} & & & \mathbf{J4} &
\end{array}\right]$$

$n \neq k$

$$\mathrm{J1}_{kn}=\frac{\partial \mathrm{P}_k}{\partial \delta_n}=\mathrm{V}_k\mathrm{Y}_{kn}\mathrm{V}_n\sin(\delta_k-\delta_n-\theta_{kn})$$

$$\mathrm{J2}_{kn}=\frac{\partial \mathrm{P}_k}{\partial \mathrm{V}_n}=\mathrm{V}_k\mathrm{Y}_{kn}\cos(\delta_k-\delta_n-\theta_{kn})$$

$$\mathrm{J3}_{kn}=\frac{\partial \mathrm{Q}_k}{\partial \delta_n}=-\mathrm{V}_k\mathrm{Y}_{kn}\mathrm{V}_n\cos(\delta_k-\delta_n-\theta_{kn})$$

$$\mathrm{J4}_{kn}=\frac{\partial \mathrm{Q}_k}{\partial \mathrm{V}_n}=\mathrm{V}_k\mathrm{Y}_{kn}\sin(\delta_k-\delta_n-\theta_{kn})$$

$n = k$

$$\mathrm{J1}_{kk}=\frac{\partial \mathrm{P}_k}{\partial \delta_k}=-\mathrm{V}_k\sum_{\substack{n-1\\ n\neq k}}^{N}\mathrm{Y}_{kn}\mathrm{V}_n\sin(\delta_k-\delta_n-\theta_{kn})$$

$$\mathrm{J2}_{kk}=\frac{\partial \mathrm{P}_k}{\partial \mathrm{V}_k}=\mathrm{V}_k\mathrm{Y}_{kk}\cos\theta_{kk}+\sum_{n-1}^{N}\mathrm{Y}_{kn}\mathrm{V}_n\cos(\delta_k-\delta_n-\theta_{kn})$$

$$\mathrm{J3}_{kk}=\frac{\partial \mathrm{Q}_k}{\partial \delta_k}=\mathrm{V}_k\sum_{\substack{n-1\\ n\neq k}}^{N}\mathrm{Y}_{kn}\mathrm{V}_n\cos(\delta_k-\delta_n-\theta_{kn})$$

$$\mathrm{J4}_{kk}=\frac{\partial \mathrm{Q}_k}{\partial \mathrm{V}_k}=-\mathrm{V}_k\mathrm{Y}_{kk}\sin\theta_{kk}+\sum_{n-1}^{N}\mathrm{Y}_{kn}\mathrm{V}_n\sin(\delta_k-\delta_n-\theta_{kn})$$

$$k,n=2,3,\ldots,N$$

Applying NR method:

STEP 1 Use (6.6.2) and (6.6.3) to compute

$$\Delta \mathbf{y}(i) = \begin{bmatrix} \Delta \mathbf{P}(i) \\ \Delta \mathbf{Q}(i) \end{bmatrix} = \begin{bmatrix} \mathbf{P} - \mathbf{P}[\mathbf{x}(i)] \\ \mathbf{Q} - \mathbf{Q}[\mathbf{x}(i)] \end{bmatrix} \tag{6.6.5}$$

STEP 2 Use the equations in Table 6.5 to calculate the Jacobian matrix.

STEP 3 Use Gauss elimination and back substitution to solve

$$\left[\begin{array}{c|c} \mathbf{J1}(i) & \mathbf{J2}(i) \\ \hline \mathbf{J3}(i) & \mathbf{J4}(i) \end{array}\right] \begin{bmatrix} \Delta \boldsymbol{\delta}(i) \\ \Delta \mathbf{V}(i) \end{bmatrix} = \begin{bmatrix} \Delta \mathbf{P}(i) \\ \Delta \mathbf{Q}(i) \end{bmatrix} \tag{6.6.6}$$

STEP 4 Compute

$$\mathbf{x}(i+1) = \begin{bmatrix} \boldsymbol{\delta}(i+1) \\ \mathbf{V}(i+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\delta}(i) \\ \mathbf{V}(i) \end{bmatrix} + \begin{bmatrix} \Delta \boldsymbol{\delta}(i) \\ \Delta \mathbf{V}(i) \end{bmatrix} \tag{6.6.7}$$

For the voltage-controlled buss, its V_k is omited from x
and $Q_k(x)$ " " from y.

Read Ex 6.11
Solve Ex 6.12
Solve Ex 6.13